OCR 4726 June 2009 Question 6

If $\int_0^1 \frac{1}{\sqrt{16+9x^2}}\,dx + \int_0^2 \frac{1}{\sqrt{9+4x^2}}\,dx = \ln a$, find a exactly.

Let $x = \frac{4}{3}\tan\theta$

$$\frac{dx}{d\theta} = \frac{4}{3}\sec^2\theta$$
$$dx = \frac{4}{3}\sec^2\theta\, d\theta$$

When $x = 0$

$$0 = \frac{4}{3}\tan\theta$$
$$\theta = 0$$

When $x = 1$

$$1 = \frac{4}{3}\tan\theta$$
$$\theta = \tan^{-1}\left(\frac{3}{4}\right)$$

Let $x = \frac{3}{2}\tan\theta$

$$\frac{dx}{d\theta} = \frac{3}{2}\sec^2\theta$$
$$dx = \frac{3}{2}\sec^2\theta\, d\theta$$

When $x = 0$

$$0 = \frac{3}{2}\tan\theta$$
$$\theta = 0$$

When $x = 2$

$$2 = \frac{3}{2}\tan\theta$$
$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

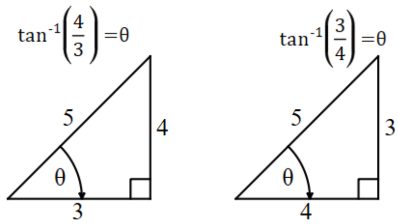
$$\int_0^{\tan^{-1}\left(\frac{3}{4}\right)} \frac{1}{\sqrt{16+9\left(\frac{16}{9}\tan^2\theta\right)}} \times \frac{4}{3}\sec^2\theta\, d\theta + \int_0^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{1}{\sqrt{9+4\left(\frac{9}{4}\tan^2\theta\right)}} \times \frac{3}{2}\sec^2\theta\, d\theta$$

$$\int_0^{\tan^{-1}\left(\frac{3}{4}\right)} \frac{1}{\sqrt{16+16\tan^2\theta}} \times \frac{4}{3}\sec^2\theta\, d\theta + \int_0^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{1}{\sqrt{9+9\tan^2\theta}} \times \frac{3}{2}\sec^2\theta\, d\theta$$

$$\int_0^{\tan^{-1}\left(\frac{3}{4}\right)} \frac{1}{4\sec\theta} \times \frac{4}{3}\sec^2\theta\, d\theta + \int_0^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{1}{3\sec\theta} \times \frac{3}{2}\sec^2\theta\, d\theta$$

$$\int_0^{\tan^{-1}\left(\frac{3}{4}\right)} \frac{\sec\theta}{3}\, d\theta + \int_0^{\tan^{-1}\left(\frac{4}{3}\right)} \frac{\sec\theta}{2}\, d\theta$$

$$\frac{1}{3}\left(\ln|\sec\theta + \tan\theta|\right)\Big]_0^{\tan^{-1}\left(\frac{3}{4}\right)} + \frac{1}{2}\left(\ln|\sec\theta + \tan\theta|\right)\Big]_0^{\tan^{-1}\left(\frac{4}{3}\right)}$$



$$\frac{1}{3}\left(\ln\left|\frac{5}{4}+\frac{3}{4}\right| - \ln|1-0|\right) + \frac{1}{2}\left(\ln\left|\frac{5}{3}+\frac{4}{3}\right| - \ln|1-0|\right)$$
$$\frac{1}{3}\ln 2 + \frac{1}{2}\ln 3$$
$$\ln a = \ln\left(2^{\frac{1}{3}} \times 3^{\frac{1}{2}}\right)$$

$$a = 2^{\frac{1}{3}} \times 3^{\frac{1}{2}}$$