The Point P in an Argand diagram represents the complex number z, which satisfies

$$\arg\left(\frac{z-1-i}{z-2i}\right)=\frac{\pi}{3}, \qquad z\neq 2i$$

It is further given that P lies on the arc AB of a circle centred at C and of radius r.

a) Sketch in an Argand diagram the circular arc AB, stating the co-ordinates of C and the value of r.
b) Given that $|PA|=|PB|$, find the complex number represented by P.

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a) Let $z=x+yi$

$$\arg\left(\frac{x+yi-1-i}{x+yi-2i}\right)=\frac{\pi}{3}$$
$$\arg\left(\frac{x-1+(y-1)i}{x+(y-2)i}\right)=\frac{\pi}{3}$$
$$\arg\left(\frac{x-1+(y-1)i}{x+(y-2)i}\times\frac{x-(y-2)i}{x-(y-i)i}\right)=\frac{\pi}{3}$$
$$\arg\left(\frac{x(x-1)-(x-1)(y-2)i+x(y-1)i+(y-1)(y-2)}{x^2+(y-2)^2}\right)=\frac{\pi}{3}$$
$$\arg\left(\frac{x(x-1)+(y-1)(y-2)}{x^2+(y-2)^2}+\frac{-(x-1)(y-2)i+x(y-1)i}{x^2+(y-2)^2}\right)=\frac{\pi}{3}$$

Because the argument is $\frac{\pi}{3}$ the real part and the imaginary part must both be greater than zero, and the tangent is $\sqrt{3}$

$$\frac{-(x-1)(y-2)+x(y-1)}{x(x-1)+(y-1)(y-2)}=\sqrt{3}$$
$$\frac{-(xy-2x-y+2)+xy-x}{x(x-1)+(y-1)(y-2)}=\sqrt{3}$$
$$\frac{-xy+2x+y-2+xy-x}{x(x-1)+(y-1)(y-2)}=\sqrt{3}$$
$$\frac{x+y-2}{x^2-x+y^2-3y+2}=\sqrt{3}$$
$$x+y-2=\sqrt{3}(x^2-x+y^2-3y+2)$$
$$0=\sqrt{3}x^2-(\sqrt{3}+1)x+\sqrt{3}y^2-(3\sqrt{3}+1)y+2\sqrt{3}+2$$
$$0=\sqrt{3}\left(x^2-\left(\frac{\sqrt{3}+1}{\sqrt{3}}\right)x\right)+\sqrt{3}\left(y^2-\left(\frac{3\sqrt{3}+1}{\sqrt{3}}\right)\right)+2\sqrt{3}+2$$
$$0=\sqrt{3}\left(\left(x-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)^2\right)+\sqrt{3}\left(\left(y-\left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2-\left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)^2\right)+\sqrt{3}\left(\frac{2\sqrt{3}+2}{\sqrt{3}}\right)$$
$$0=\left(x-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)^2+\left(y-\left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2-\left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)^2+\sqrt{3}\left(\frac{2\sqrt{3}+2}{\sqrt{3}}\right)$$
$$0=\left(x-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2+\left(y-\left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2-\frac{2}{3}$$
$$\frac{2}{3}=\left(x-\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2+\left(y-\left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2$$

Hence the circle has a centre at $\left(\frac{\sqrt{3}+1}{2\sqrt{3}}, \frac{3\sqrt{3}+1}{2\sqrt{3}}\right)$ and radius is $\sqrt{\frac{2}{3}}$

But which part of the circle is our arc AB?

We know

$$x + y - 2 > 0$$

$$\therefore y > -x + 2$$

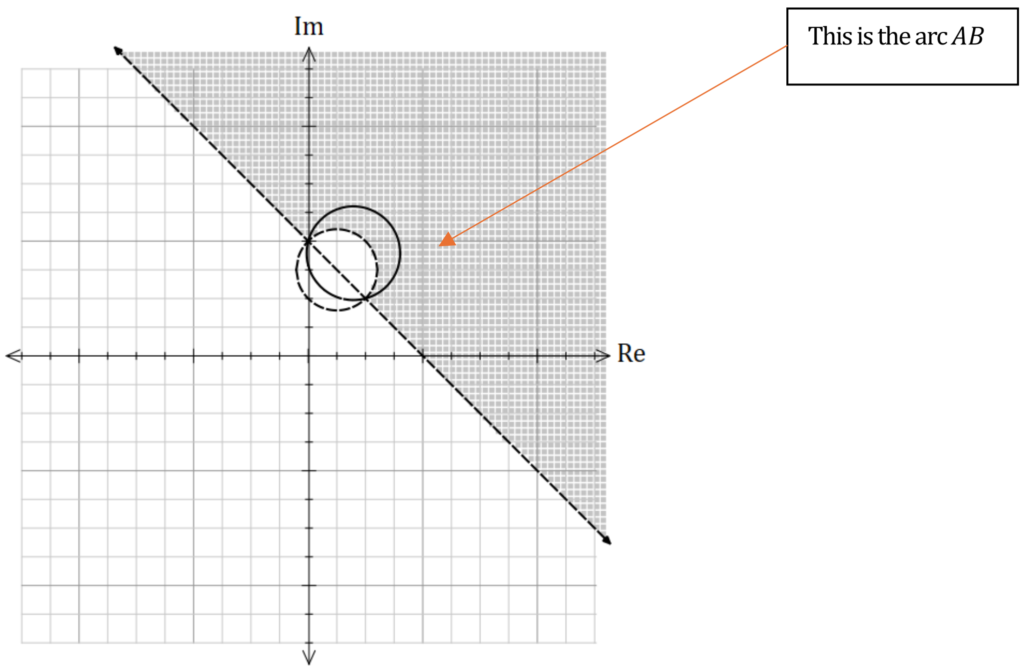
And

$$x^2 - x + y^2 - 3y + 2 > 0$$

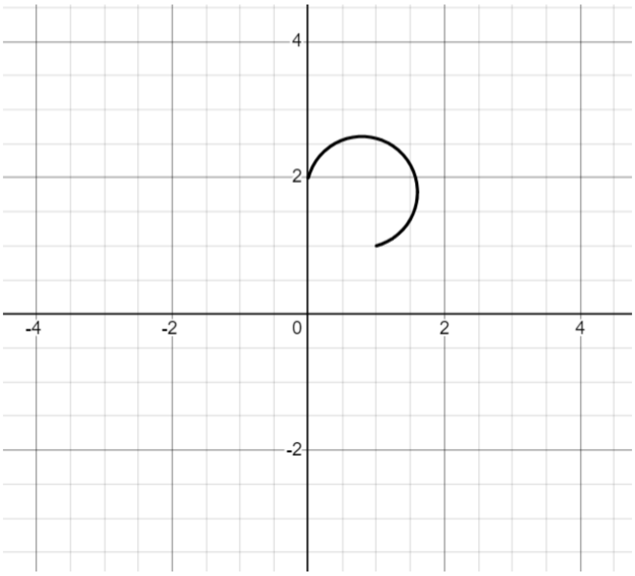
$$\left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \left(y - \frac{3}{2}\right)^2 - \frac{9}{4} + 2 > 0$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 - \frac{1}{2} > 0$$

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 > \frac{1}{2}$$



Hence, this is the circular arc AB



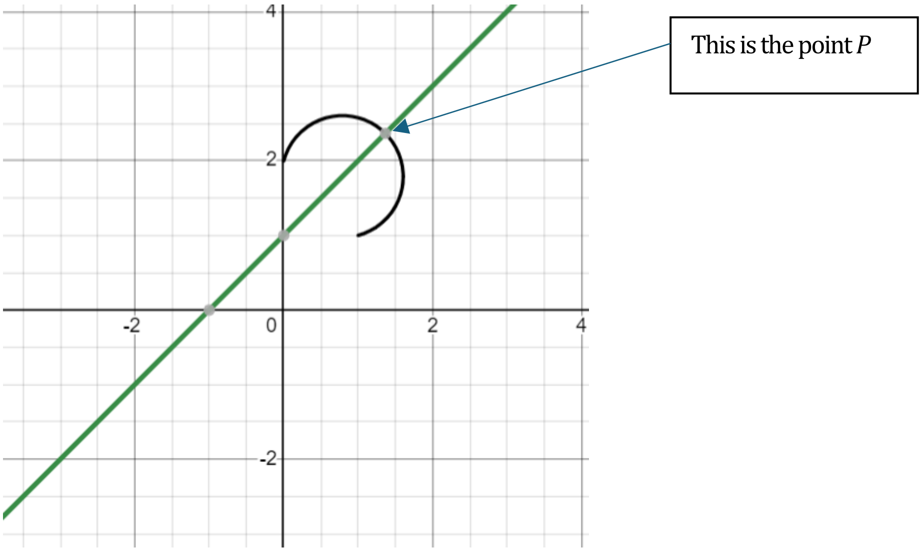
b) A and B are the endpoints of the arc,

$A = (0.2)$ and $B = (1, 1)$

P is equidistant to A and B

Let $P = (x, y)$

$$\sqrt{(0-x)^2 + (2-y)^2} = \sqrt{(1-x)^2 + (y-1)^2}$$
$$x^2 + 4 - 4y + y^2 = 1 - 2x + x^2 + y^2 - 2y + 1$$
$$0 = -2x - 2 + 2y$$
$$0 = -x - 1 + y$$
$$x + 1 = y$$
$$y = x + 1$$



To find the co-ordinates of P we need to substitute $y = x + 1$ into the circle equation.

$$\frac{2}{3} = \left(x - \left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2 + \left(y - \left(\frac{3\sqrt{3}+1}{2\sqrt{3}}\right)\right)^2$$

$$\frac{\sqrt{3}+1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3+\sqrt{3}}{6} = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right)$$

$$\frac{3\sqrt{3}+1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{9+\sqrt{3}}{6} = \frac{1}{2}\left(3 + \frac{\sqrt{3}}{3}\right)$$

$$\frac{2}{3} = \left(x - \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right)\right)^2 + \left(x + 1 - \frac{1}{2}\left(3 + \frac{\sqrt{3}}{3}\right)\right)^2$$

$$\frac{2}{3} = \left(x - \frac{1}{2} - \frac{\sqrt{3}}{6}\right)^2 + \left(x - \frac{1}{2} - \frac{\sqrt{3}}{6}\right)^2$$

$$\frac{2}{3} = 2\left(x - \frac{1}{2} - \frac{\sqrt{3}}{6}\right)^2$$

$$\frac{1}{3} = \left(x - \frac{1}{2} - \frac{\sqrt{3}}{6}\right)^2$$

$$x - \frac{1}{2} - \frac{\sqrt{3}}{6} = \pm\frac{1}{\sqrt{3}}$$

$$x = \frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{\sqrt{3}}{6} \text{ or } x = -\frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{\sqrt{3}}{6}$$

$$x = \frac{\sqrt{3}}{3} + \frac{1}{2} + \frac{\sqrt{3}}{6} \text{ or } x = -\frac{1}{\sqrt{3}} + \frac{1}{2} + \frac{\sqrt{3}}{6}$$

$$x = \frac{2\sqrt{3} + 3 + \sqrt{3}}{6} \text{ or } x = \frac{-2\sqrt{3} + 3 + \sqrt{3}}{6}$$

$$x = \frac{1}{2} + \frac{\sqrt{3}}{2} \text{ or } x = -\frac{\sqrt{3}}{6} + \frac{1}{2}$$

We can ignore the second x value

Hence P is the point $\frac{1}{2} + \frac{\sqrt{3}}{2} + \left(\frac{3}{2} + \frac{\sqrt{3}}{2}\right)i$