Describe the locus defined by $|z-2|-|z+2|=3$

$$|z-2|-|z+2|=3$$

Let $z=x+yi$

$$|x+yi-2|-|x+yi+2|=3$$

$$|x-2+yi|-|x+2+yi|=3$$

$$\sqrt{(x-2)^2+y^2}-\sqrt{(x+2)^2+y^2}=3$$

$$\sqrt{(x-2)^2+y^2}=3+\sqrt{(x+2)^2+y^2}$$

$$(x-2)^2+y^2=\left(3+\sqrt{(x+2)^2+y^2}\right)^2$$

$$(x-2)^2+y^2=9+6\sqrt{(x+2)^2+y^2}+(x+2)^2+y^2$$

$$(x-2)^2+y^2-(x+2)^2-y^2-9=6\sqrt{(x+2)^2+y^2}$$

$$x^2-4x+4-(x^2+4x+4)-9=6\sqrt{(x+2)^2+y^2}$$

$$-8x-9=6\sqrt{(x+2)^2+y^2}$$

$$(-8x-9)^2=36((x+2)^2+y^2)$$

$$64x^2+144x+81=36(x^2+4x+4)+36y^2$$

$$64x^2+144x+81-36x^2-144x-144-36y^2=0$$

$$28x^2-36y^2=63$$

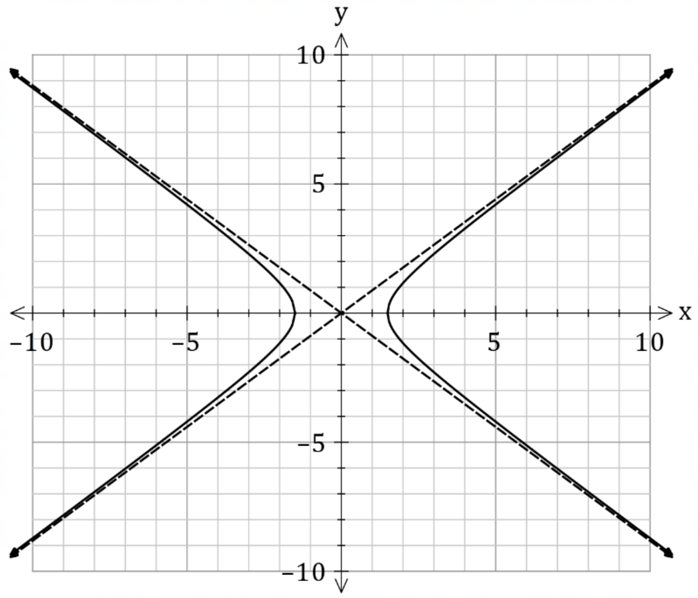
$$x^2-\frac{36y^2}{28}=\frac{63}{28}$$

$$x^2-\frac{9y^2}{7}=\frac{9}{4}$$

$$\frac{x^2}{9}-\frac{y^2}{7}=\frac{9}{36}$$

$$\frac{x^2}{9}-\frac{y^2}{7}=\frac{1}{4}$$

This is the Cartesian equation of a hyperbola with asymptotes $y=\pm\frac{\sqrt{7}}{3}x$



We know from

$$-8x - 9 = 6\sqrt{(x+2)^2 + y^2}$$

That

$-8x - 9 \geq 0$

Hence,

$$-8x \geq 9$$

$$x \leq -\frac{9}{8}$$

Therefore, the locus is the left branch of this hyperbola.